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THE INTERSECTIONS OF A STRAIGHT LINE AND HYPERQUADRIC.*

BY G. M. GREEN.

In a recent note,† Coolidge obtained explicit expressions for the coördinates of the points of intersection of a line and a hyperquadric in space of n dimensions. In the present note we obtain Coolidge's formula by a simple method, suggested by geometric considerations. In fact, our discussion makes clear the geometric significance of all of Coolidge's symbols, including his undetermined multipliers $z_1, z_2, \cdots, z_n$.

Let Q be a hyperquadric, and l a line, in n-space. The line l cuts the quadric in two points M and N whose coördinates we wish to determine. Let α be a point of the line l not on the quadric, and let β be the harmonic conjugate of α with respect to M and N. In homogeneous coördinates the equation of the quadric Q may be written

$$\sum_{i,j} a_{ij}x_ix_j = 0, \qquad a_{ij} = a_{ji}. \tag{1}$$

In this, as in all other summations to follow, the indicated indices are to run independently from 0 to n inclusive. If the coördinates of α and β are $\alpha_0, \alpha_1, \cdots, \alpha_n$ and $\beta_0, \beta_1, \cdots, \beta_n$, the coördinates of any point of the line l will be of the form $\beta_i + p\alpha_i$ $(i = 0, 1, \cdots, n)$. We wish to determine p so that the point shall lie on the quadric. Substituting its coördinates in equation (1), we find that the equation for p is the quadratic

$$p^2 \sum_{i,j} a_{ij}\alpha_i\alpha_j + \sum_{i,j} a_{ij}\beta_i\beta_j = 0, \tag{2}$$

from which the linear term is absent because the points α, β separate harmonically the points M, N. It remains merely to obtain suitable expressions for the quantities α and β in the above formulas.

With Coolidge, let us define the line l by the $n - 1$ linear equations

$$\sum_i u_i^{(1)}x_i = 0, \qquad \sum_i u_i^{(2)}x_i = 0, \qquad \cdots, \qquad \sum_i u_i^{(n-1)}x_i = 0, \tag{3}$$

and let us write

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[*] Presented to the American Mathematical Society, April 28, 1917.

[†] Cf. these *Annals*, Series 2, vol. 18 (1916–17), pp. 209–13.

$$\Delta = \begin{vmatrix} v_0, & v_1, & \cdots, & v_n \\ w_0, & w_1, & \cdots, & w_n \\ u_0^{(1)}, & u_1^{(1)}, & \cdots, & u_n^{(1)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ u_0^{(n-1)}, & u_1^{(n-1)}, & \cdots, & u_n^{(n-1)} \end{vmatrix}, \tag{4}$$

$$p_{ij} = - p_{ji} = \frac{1}{\Delta}\left(\frac{\partial \Delta}{\partial v_i}\frac{\partial \Delta}{\partial w_j} - \frac{\partial \Delta}{\partial v_j}\frac{\partial \Delta}{\partial w_i}\right). \tag{5}$$

The quantity p_{ij} is obviously, except possibly for sign, the $(n-1)$-rowed determinant formed from the last $n-1$ rows of Δ by deleting the columns with lower indices i and j.

Using expressions of the form (5), which are merely generalized Plücker coördinates of the line l, we may easily establish the identity

$$p_{ij}p_{kl} + p_{ik}p_{lj} + p_{il}p_{jk} = 0, \tag{6}$$

exactly as is done in the case of the identity connecting the Plücker coördinates in ordinary space.

The point α on the line l may now be determined by adjoining to equations (3) another linear equation,

$$\sum_i z_i x_i = 0, \tag{7}$$

which is linearly independent of equations (3). Solving equations (3) and (7) for the ratios of the x's, we obtain for the coördinates α_i of the point α the expressions

$$\alpha_i = \sum_j z_j p_{ij} \qquad (i = 0, 1, \cdots, n). \tag{8}$$

Since the point α is not on the quadric,

$$\sum_{i,j} a_{ij}\alpha_i\alpha_j \neq 0. \tag{9}$$

The polar of α with respect to the quadric is the hyperplane

$$\sum_{i,j} a_{ij}\alpha_i x_j = 0, \tag{10}$$

and the intersection of this with the line l gives the point β, whose coördinates are found by solving equations (3) and (10) for the x's, and are therefore

$$\beta_i = \sum_{j,k} a_{jk}\alpha_k p_{ij} \qquad (i = 0, 1, \cdots, n). \tag{11}$$

In virtue of (8), we may write

$$\beta_i = \sum_{j,\,k,\,l} a_{jk} z_l p_{ij} p_{kl} \qquad (i = 0,\, 1,\, \cdots,\, n). \tag{12}$$

Equations (11) and (12) being written in the form

$$\beta_i = \sum_{k,\,l} a_{kl} \alpha_l p_{ik}, \qquad \beta_j = \sum_{r,\,s,\,t} a_{rs} z_t p_{jr} p_{st}, \tag{13}$$

we have

$$\sum a_{ij} \beta_i \beta_j = \sum a_{ij} a_{kl} a_{rs} \alpha_l z_t p_{ik} p_{st} p_{jr}, \tag{14}$$

in which the summations are taken with respect to all the subscripts. By (6), we may put

$$p_{ik} p_{st} = p_{is} p_{kt} - p_{it} p_{ks},$$

so that (14) becomes

$$\sum a_{ij} \beta_i \beta_j = \sum a_{ij} a_{kl} a_{rs} \alpha_l z_t p_{is} p_{kt} p_{jr} - \sum a_{ij} a_{kl} a_{rs} \alpha_l z_t p_{it} p_{ks} p_{jr}. \tag{15}$$

In the right-hand member, the second summation obviously becomes the same as the right-hand member of (14), if the indices i and s are interchanged, and likewise the indices j and r.* Therefore, transposing the second term of the right-hand member of (15) to the other side, and rearranging the first term slightly, we obtain the equation

$$2 \sum a_{ij} \beta_i \beta_j = \sum a_{ij} a_{rs} p_{is} p_{jr} (a_{kl} \alpha_l z_t p_{kt}). \tag{16}$$

The indices k, l, t appear only in the part enclosed in parentheses, therefore that part may be taken out as a factor† $\sum a_{kl} \alpha_l z_t p_{kt}$. But by (8) this factor may be written $\Sigma\, a_{kl} \alpha_k \alpha_l$, so that equation (2) finally becomes, in virtue of (9),

$$p^2 + \tfrac{1}{2} \sum a_{ij} a_{rs} p_{is} p_{jr} = 0.$$

We may therefore state the final result as follows:

The line l intersects the hyperquadric Q in two points whose coördinates are given by

$$x_i = \beta_i + p\alpha_i,$$

where

$$\alpha_i = \sum_j z_j p_{ij}, \qquad \beta_i = \sum_{j,\,k\,l} z_l a_{jk} p_{ij} p_{kl},$$

$$p^2 + \tfrac{1}{2} \sum_{i,\,j,\,k,\,l} a_{ij} a_{kl} p_{il} p_{jk} = 0.$$

HARVARD UNIVERSITY,
April, 1917.

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* It should be remembered that by (5) $p_{ij} = -p_{ji}$, and also that $a_{ij} = a_{ji}$.

† This factorization of $\Sigma a_{ij} \beta_i \beta_j$ yields a very general theorem on determinants, only special cases of which the writer is aware of in the literature.